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FEYNMAN–KAC FORMULA FOR THE HEAT EQUATION DRIVEN BY FRACTIONAL NOISE WITH HURST PARAMETER $H < 1/2$

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In this paper, a Feynman–Kac formula is established for stochastic partial differential equation driven by Gaussian noise which is, with respect to time, a fractional Brownian motion with Hurst parameter $H < 1/2$. To establish such a formula, we introduce and study a nonlinear stochastic integral from the given Gaussian noise. To show the Feynman–Kac integral exists, one still needs to show the exponential integrability of nonlinear stochastic integral. Then, the approach of approximation with techniques from Malliavin calculus is used to show that the Feynman–Kac integral is the weak solution to the stochastic partial differential equation.

1. Introduction.

Consider the stochastic heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + u\,\dfrac{\partial W}{\partial t}(t,x), \qquad t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where u_0 is a bounded measurable function and $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in time and it has a spatial covariance $Q(x,y)$, which is locally γ-Hölder continuous (see Section 2 for precise meaning of this condition), with $\gamma > 2 - 4H$. We shall show that the solution to (1.1) is given by

$$u(t,x) = E^B\left[u_0(B_t^x)\exp\int_0^t W(ds, B_{t-s}^x)\right], \tag{1.2}$$

where $B = \{B_t^x = B_t + x, t \geq 0, x \in \mathbb{R}^d\}$ is a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$, independent of W.

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This is a generalization of the well-known Feyman–Kac formula to the case of a random potential of the form $\frac{\partial W}{\partial t}(t,x)$. Notice that the integral $\int_0^t W(ds, B_{t-s}^x)$ is a nonlinear stochastic integral with respect to the fractional noise W. This type of Feynman–Kac formula was mentioned as a conjecture by Mocioalca and Viens in [6].

There exists an extensive literature devoted to Feynman–Kac formulas for stochastic partial differential equations. Different versions of the Feynman–Kac formula have been established for a variety of random potentials. See, for instance, a Feynman–Kac formula for anticipating SPDE proved by Ocone and Pardoux [9]. Ouerdiane and Silva [10] give a generalized Feynman–Kac formula with a convolution potential by introducing a generalized function space. Feynman–Kac formulas for Lévy processes are presented by Nualart and Schoutens [8].

However, only recently a Feynman–Kac formula has been established by Hu et al. [4] for random potentials associated with the fractional Brownian motion. The authors consider the following stochastic heat equation driven by fractional noise

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + u\,\dfrac{\partial^{d+1} W}{\partial t\,\partial x_1 \cdots \partial x_d}(t,x), & t \ge 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

where $W = \{W(t,x), t \ge 0, x \in \mathbb{R}^d\}$ is fractional Brownian sheet with Hurst parameter $(H_0, H_1, \ldots, H_d)$. They show ([4], Theorem 4.3) that if $H_1, \ldots, H_d \in (\frac{1}{2}, 1)$, and $2H_0 + H_1 + \cdots + H_d > d+1$, then the solution $u(t,x)$ to the above stochastic heat equation is given by

$$u(t,x) = E^B\left[f(B_t^x)\exp\left(\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)\right)\right], \tag{1.4}$$

where $B = \{B_t^x = B_t + x, t \ge 0, x \in \mathbb{R}^d\}$ is a d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$, independent of W. The condition $2H_0 + H_1 + \cdots + H_d > d+1$ is shown to be sharp in that framework. Since the H_i, $i = 1, \ldots, d$, cannot take value greater or equal to 1, this condition implies that $H_0 > \frac{1}{2}$.

We remark that if $B^{H_0} = \{B_t^{H_0}, t \ge 0\}$ is a fractional Brownian motion with Hurst parameter $H_0 > \frac{1}{2}$, then the stochastic integral $\int_0^T f(t)\,dB_t^{H_0}$ is well defined for a suitable class of distributions f, and in this sense the above integral $\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)$ is well defined for any trajectory of the Brownian motion B. If $H_0 < \frac{1}{2}$, this is no longer true and we can integrate only functions satisfying some regularity conditions. For this reason, it is not possible to write a Feynman–Kac formula for the equation (1.3) with $H_0 < \frac{1}{2}$.

Notice that for $d = 1$ and $H_0 = H_1 = \frac{1}{2}$ (space–time white noise) a Feynman–Kac formula can not be written for equation (1.3), but this equation

has a unique mild solution when the stochastic integral is interpreted in the Itô sense. A renormalized Feynman–Kac formula with Wick exponential has been obtained in this case by Bertinin and Cancrini [1]. More generally, if the product appearing in (1.3) is replaced by Wick product, Hu and Nualart [3] showed that a formal solution can be obtained using chaos expansions.

In the present paper, we are concerned with the case $H_0 < \frac{1}{2}$, but we use a random potential of the form $\frac{\partial W}{\partial t}(t,x)$. One of the main obstacles to overcome is to define the stochastic integral $\int_0^t W(ds, B_{t-s}^x)$. We start with the construction of a general nonlinear stochastic integral $\int_0^t W(ds,\phi_s)$ where ϕ is a Hölder continuous function of order $\alpha > \frac{1}{\gamma}(1-2H)$. It turns out that the irregularity in time of $W(t,x)$ is compensated by the above Hölder continuity of ϕ through the covariance in space, with an appropriate application of the fractional integration by parts technique. Let us point out that $\int_0^t W(ds,\phi_s)$ is well defined for all Hölder continuous function ϕ with $\alpha > \frac{1}{\gamma}(\frac{1}{2}-H)$, and we consider here only the case $\alpha > \frac{1}{\gamma}(1-2H)$ because this condition is required when we show that $u(t,x)$ is a weak solution to (1.1). Furthermore, the condition $\alpha > \frac{1}{\gamma}(1-2H)$ also allows us to obtain an explicit formula for the variance of $\int_0^t W(ds,\phi_s)$. Contrary to [4], it is rather simpler to show that $\int_0^t W(ds, B_{t-s}^x)$ is exponentially integrable. A by-product is that $u(t,x)$ defined by (1.2) is almost surely Hölder continuous of order which can be arbitrarily close to $H - \frac{1}{2} + \frac{\gamma}{4}$ from below. Let us also mention recent work on stochastic integral [2] and [5] with general Gaussian processes which can be applied to the case $H < \frac{1}{2}$.

Another main effort of this paper is to show that $u(t,x)$ defined by (1.2) is a solution to (1.1) in a weak sense (see Definition 5.2). As in [4], this is done by using an approximation scheme together with techniques of Malliavin calculus. Let us point out that in the definition of $\int_0^t W(ds,\phi_s)$ one can use a one-side approximation, but it is necessary to use symmetric approximations (as well as the condition $H > \frac{1}{2} - \frac{\gamma}{4}$) to show the convergence of the trace term (5.10).

We also discuss the corresponding Skorohod-type equation, which corresponds to taking the Wick product in [3]. We show that a unique mild solution exists for $H \in (\frac{1}{2} - \frac{\gamma}{4}, \frac{1}{2})$.

The paper is organized as follows. Section 2 contains some preliminaries on the fractional noise W and some results on fractional calculus which is needed in the paper. We also list all the assumptions that we make for the noise W in this section. In Section 3, we study the nonlinear stochastic integral appeared in equation (1.2) by using smooth approximation and we derive some basic properties of this integral. Section 4 verifies the integrability and Hölder continuity of $u(t,x)$. Section 5 is devoted to show that $u(t,x)$ is a solution to (1.1) in a weak sense. Section 6 gives a solution to

the Skorohod type equation. The last section is the Appendix with some technical results used along the paper.

2. Preliminaries. Fix $H \in (0, \frac{1}{2})$ and denote by $R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ the covariance function of the fractional Brownian motion of Hurst parameter H. Suppose that $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mean zero Gaussian random field, defined on a probability space $(\Omega, \mathcal{F}, P)$, whose covariance function is given by

$$E(W(t,x)W(s,y)) = R_H(t,s)Q(x,y),$$

where $Q(x,y)$ satisfies the following properties for some $M < 2$ and $\gamma \in (0,1]$:

(Q1) Q is locally bounded: there exists a constant $C_0 > 0$ such that for any $K > 0$

$$Q(x,y) \leq C_0(1+K)^M$$

for any $x, y \in \mathbb{R}^d$ such that $|x|, |y| \leq K$.

(Q2) Q is locally γ-Hölder continuous: there exists a constant $C_1 > 0$ such that for any $K > 0$

$$|Q(x,y) - Q(u,v)| \leq C_1(1+K)^M(|x-u|^\gamma + |y-v|^\gamma)$$

for any $x, y, u, v \in \mathbb{R}^d$ such that $|x|, |y|, |u|, |v| \leq K$.

Denote by $\mathcal{E}$ the vector space of all step functions on $[0,T]$. On this vector space $\mathcal{E}$, we introduce the following scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}_0} = R_H(t,s).$$

Let $\mathcal{H}_0$ be the closure of $\mathcal{E}$ with respect to the above scalar product. Denote by $C^\alpha([a,b])$ the set of all functions which is Hölder continuous of order α, and denote by $\|\cdot\|_\alpha$ the α-Hölder norm. It is well known that $C^\alpha([0,T]) \subset \mathcal{H}_0$ for $\alpha > \frac{1}{2} - H$.

Let $\mathcal{H}$ be the Hilbert space defined by the completion of the linear span of indicator functions $\mathbf{1}_{[0,t]\times[0,x]}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ under the scalar product

$$\langle \mathbf{1}_{[0,t]\times[0,x]}, \mathbf{1}_{[0,s]\times[0,y]} \rangle_{\mathcal{H}} = R_H(t,s)Q(x,y).$$

In the above formula, if $x_i < 0$ we assume by convention that $\mathbf{1}_{[0,x_i]} = -\mathbf{1}_{[-x_i,0]}$. The mapping $W : \mathbf{1}_{[0,t]\times[0,x]} \to W(t,x)$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. Then, $\{W(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process.

Let $\mathcal{S}$ be the space of random variables F of the form:

$$F = f(W(\varphi_1), \ldots, W(\varphi_n)),$$

where $\varphi_i \in \mathcal{H}$, $f \in C^\infty(\mathbb{R}^n)$, f and all its partial derivatives have polynomial growth. The Malliavin derivative DF of an element F in $\mathcal{S}$ is defined as an

$\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(\varphi_1), \dots, W(\varphi_n))\varphi_i.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega, \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of $\mathcal{S}$ with respect to the following norm:

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}}^2)}.$$

The divergence operator δ is the adjoint of the derivative operator D, determined by the duality relationship

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathcal{H}}) \qquad \text{for any } F \in \mathbb{D}^{1,2}.$$

$\delta(u)$ is also called the Skorohod integral of u. We refer to Nualart [7] for a detailed account on the Malliavin calculus. For any random variable $F \in \mathbb{D}^{1,2}$ and $\phi \in \mathcal{H}$,

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathcal{H}}. \tag{2.1}$$

Since we deal with the case of Hurst parameter $H \in (0, 1/2)$, we shall use intensively the fractional calculus. We recall some basic definitions and properties. For a detailed account, we refer to [11].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a, b)$ and $\alpha > 0$. The left and right-sided fractional integral of f of order α are defined for $x \in (a, b)$, respectively, as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, dy.$$

Let $I_{a+}^{\alpha}(L^p)$ [resp., $I_{b-}^{\alpha}(L^p)$] the image of $L^p(a, b)$ by the operator I_{a+}^{α} (resp., I_{b-}^{α}).

If $f \in I_{a+}^{\alpha}(L^p)$ [resp., $I_{b-}^{\alpha}(L^p)$] and $0 < \alpha < 1$ then the left and right-sided fractional derivatives are defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}}\, dy \right) \tag{2.2}$$

and

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}}\, dy \right) \tag{2.3}$$

for all $x \in (a, b)$ [the convergence of the integrals at the singularity $y = x$ holds point-wise for almost all $x \in (a, b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$].

It is easy to check that if $f \in I^1_{a+(b-)}(L^1)$,

$$D^{\alpha}_{a+} D^{1-\alpha}_{a+} f = Df, \qquad D^{\alpha}_{b-} D^{1-\alpha}_{b-} f = Df \tag{2.4}$$

and

$$(-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(x) g(x)\, dx = \int_a^b f(x) D^{\alpha}_{b-} g(x)\, dx \tag{2.5}$$

provided that $0 \le \alpha \le 1$, $f \in I^{\alpha}_{a+}(L^p)$ and $g \in I^{\alpha}_{b-}(L^q)$ with $p \ge 1, q \ge 1, \frac{1}{p} + \frac{1}{q} \le 1 + \alpha$.

It is clear that $D^{\alpha} f$ exists for all $f \in C^{\beta}([a,b])$ if $\alpha < \beta$. The following proposition was proved in [12].

Proposition 2.1. *Suppose that $f \in C^{\lambda}([a,b])$ and $g \in C^{\mu}([a,b])$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann–Stieltjes integral $\int_a^b f\, dg$ exists and it can be expressed as*

$$\int_a^b f\, dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t) D^{1-\alpha}_{b-} g_{b-}(t)\, dt, \tag{2.6}$$

where $g_{b-}(t) = g(t) - g(b)$.

3. Nonlinear stochastic integral. In this section, we introduce the nonlinear stochastic integral that appears in the Feynman–Kac formula (1.2) and obtain some properties of this integral which are useful in the following sections. The main idea to define this integral is to use an appropriate approximation scheme. In order to introduce our approximation, we need to extend the fractional Brownian field to $t < 0$. This can be done by defining $W = \{W(t,x), t \in \mathbb{R}, x \in \mathbb{R}^d\}$ as a mean zero Gaussian process with the following covariance

$$E[W(t,x)W(s,y)] = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) Q(x,y).$$

For any $\varepsilon > 0$, we introduce the following approximation of $W(t,x)$:

$$W^{\varepsilon}(t,x) = \int_0^t \dot{W}^{\varepsilon}(s,x)\, ds, \tag{3.1}$$

where $\dot{W}^{\varepsilon}(s,x) = \frac{1}{2\varepsilon}(W(s+\varepsilon,x) - W(s-\varepsilon,x))$.

Definition 3.1. Given a continuous function ϕ on $[0,T]$, define

$$\int_0^t W(ds, \phi_s) = \lim_{\varepsilon \to 0} \int_0^t \dot{W}^{\varepsilon}(s, \phi_s)\, ds,$$

if the limit exits in $L^2(\Omega)$.

Now we want to find conditions on ϕ such that the above limit exists in $L^2(\Omega)$. To this end, we set $I_{\varepsilon}(\phi) = \int_0^t \dot{W}^{\varepsilon}(s, \phi_s)\, ds$ and compute

$E(I_\varepsilon(\phi)I_\delta(\phi))$ for $\varepsilon, \delta > 0$. Denote

$$V_{\varepsilon,\delta}^{2H}(r) = \frac{1}{4\varepsilon\delta}(|r+\varepsilon-\delta|^{2H} - |r+\varepsilon+\delta|^{2H} - |r-\varepsilon-\delta|^{2H} + |r-\varepsilon+\delta|^{2H}).$$

Using the fact that $Q(x,y) = Q(y,x)$, we have

$$\begin{aligned} &E(I_\varepsilon(\phi)I_\delta(\phi)) \\ &\quad = \frac{1}{4\varepsilon\delta}\int_0^t\int_0^\theta Q(\phi_\theta,\phi_\eta)[|\theta-\eta+\varepsilon-\delta|^{2H} - |\theta-\eta+\delta+\varepsilon|^{2H} \\ &\qquad\qquad - |\theta-\eta-\varepsilon-\delta|^{2H} + |\theta-\eta-\varepsilon+\delta|^{2H}]\,d\eta\,d\theta. \end{aligned}$$

Making the substitution $r = \theta - \eta$ and using the notation $V_{\varepsilon,\delta}^{2H}$, we can write

$$E(I_\varepsilon(\phi)I_\delta(\phi)) = \int_0^t\int_0^\theta Q(\phi_\theta,\phi_{\theta-r})V_{\varepsilon,\delta}^{2H}(r)\,dr\,d\theta. \tag{3.2}$$

We need the following two technical lemmas.

LEMMA 3.2. *For any bounded function $\psi : [0,T] \to \mathbb{R}$, we have*

$$\left|\int_0^t \psi(s)\int_0^s V_{\varepsilon,\delta}^{2H}(r)\,dr\,ds - 2H\int_0^t \psi(s)s^{2H-1}\,ds\right| \le 4\|\psi\|_\infty(\varepsilon+\delta)^{2H}. \tag{3.3}$$

PROOF. Let $g(s) := \int_0^s |r|^{2H}\,dr$ and $f_{\varepsilon,\delta}(t) := \int_0^t \psi(s)\int_0^s V_{\varepsilon,\delta}^{2H}(r)\,dr\,ds$. Note that g'' exists everywhere except at 0 and $g''(r) = 2H\,\mathrm{sign}(r)|r|^{2H-1}$ for $r \neq 0$. Then

$$\begin{aligned} f_{\varepsilon,\delta}(t) &= \frac{1}{4\varepsilon\delta}\int_0^t \psi(s)[g(s+\varepsilon-\delta) - g(s+\varepsilon+\delta) \\ &\qquad\qquad - g(s-\varepsilon-\delta) + g(s-\varepsilon+\delta)]\,ds \\ &= \frac{1}{4}\int_{-1}^1\int_{-1}^1\int_0^t \psi(s)g''(s+\eta\varepsilon-\xi\delta)\,ds\,d\xi\,d\eta \\ &= \frac{1}{4}\int_{-1}^1\int_{-1}^1\int_0^t \psi(s)g''(s-\Delta)\,ds\,d\xi\,d\eta, \end{aligned}$$

where $\Delta = \xi\delta - \eta\varepsilon$.

Case (i): If $\Delta \le 0$, we have

$$\begin{aligned} &\left|\int_0^t \psi(s)(g''(s-\Delta) - 2Hs^{2H-1})\,ds\right| \\ &\qquad \le 2H\|\psi\|_\infty\int_0^t (s^{2H-1} - (s-\Delta)^{2H-1})\,ds \qquad (3.4) \\ &\qquad = \|\psi\|_\infty[t^{2H} - (t-\Delta)^{2H} + (-\Delta)^{2H}] \le 2\|\psi\|_\infty|\Delta|^{2H}. \end{aligned}$$

Case (ii): If $\Delta > 0$, we assume that $\Delta < t$ (the case $\Delta \geq t$ follows easily). Then

$$\int_0^t \psi(s) g''(s-\Delta)\,ds = -2H \int_0^\Delta \psi(s)(\Delta - s)^{2H-1}\,ds \\ + 2H \int_\Delta^t \psi(s)(s-\Delta)^{2H-1}\,ds.$$

Therefore,

$$\left| \int_0^t \psi(s)(g''(s-\Delta) - 2Hs^{2H-1})\,ds \right| \leq F_\Delta^1 + F_\Delta^2, \tag{3.5}$$

where

$$F_\Delta^1 := 2H \int_0^\Delta \psi(s)[(\Delta - s)^{2H-1} + s^{2H-1}]\,ds \leq 2\|\psi\|_\infty |\Delta|^{2H} \tag{3.6}$$

and

$$\begin{aligned} F_\Delta^2 &:= 2H \int_\Delta^t \psi(s)[(s-\Delta)^{2H-1} - s^{2H-1}]\,ds \\ &\leq 2H\|\psi\|_\infty \int_\Delta^t [(s-\Delta)^{2H-1} - s^{2H-1}]\,ds \leq 2\|\psi\|_\infty |\Delta|^{2H}. \end{aligned} \tag{3.7}$$

Then (3.3) follows from (3.4)–(3.7). □

LEMMA 3.3. *Let $\psi \in C([0,T]^2)$ with $\psi(0,s) = 0$, and $\psi(\cdot, s) \in C^\alpha([0,T])$ for any $s \in [0,T]$. Assume $\alpha + 2H > 1$ and $\sup_{s \in [0,T]} \|\psi(\cdot,s)\|_\alpha < \infty$. Then for any $1 - 2H < \gamma < \alpha$ and $t \leq T$ we have*

$$\begin{aligned} &\left| \int_0^t \int_0^s \psi(r,s)[V_{\varepsilon,\delta}^{2H}(r) - 2H(2H-1)r^{2H-2}]\,dr\,ds \right| \\ &\qquad \leq C \sup_{s\in[0,T]} \|\psi(\cdot,s)\|_\alpha (\varepsilon + \delta)^{2H+\gamma-1}, \end{aligned} \tag{3.8}$$

where the constant C depends on H, γ, α and T, but it is independent of δ, ε and ψ.

PROOF. Along the proof, we denote by C a generic constant which depends on H, γ, α and T. Set $h(r) := |r|^{2H}$. Then $h'(r)$ exists everywhere except at 0 and $h'(r) = 2H\,\mathrm{sign}(r)|r|^{2H-1}$ if $r \neq 0$. Using (2.4) and (2.6), we have

$$\begin{aligned} f_{\varepsilon,\delta}(t) &:= \int_0^t \int_0^s \psi(r,s) V_{\varepsilon\delta}^{2H}(r)\,dr\,ds \\ &= \frac{1}{4\varepsilon} \int_{-1}^1 \int_0^t \int_0^s \psi(r,s) \frac{\partial}{\partial r}[h(r+\varepsilon-\xi\delta) - h(r-\varepsilon-\xi\delta)]\,dr\,ds\,d\xi \end{aligned}$$

$$= (-1)^{\alpha'} \frac{1}{4\varepsilon} \int_{-1}^{1} \int_0^t \int_0^s D_{0+}^{\alpha'} \psi(r,s) \times D_{s-}^{1-\alpha'} [h(r+\varepsilon-\xi\delta) - h(r-\varepsilon-\xi\delta)] \, dr \, ds \, d\xi$$

$$= (-1)^{\alpha'} \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \int_0^t \int_0^s D_{0+}^{\alpha'} \psi(r,s) D_{s-}^{1-\alpha'} h'(r+\eta\varepsilon-\xi\delta) \, dr \, ds \, d\xi \, d\eta,$$

where $\gamma < \alpha' < \alpha$. On the other hand, we also have

$$2H(2H-1) \int_0^t \int_0^s \psi(r,s) r^{2H-2} \, dr = (-1)^{\alpha'} \int_0^t \int_0^s D_{0+}^{\alpha'} \psi(r,s) D_{s-}^{1-\alpha'} h'(r) \, dr \, ds.$$

Thus,

$$I_{\varepsilon,\delta} := \left| \int_0^t \int_0^s \psi(r,s) [V_{\varepsilon,\delta}^{2H}(r) - 2H(2H-1) r^{2H-2}] \, dr \, ds \right| \le \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \int_0^t \int_0^s |D_{0+}^{\alpha'} \psi(r,s)| \times |D_{s-}^{1-\alpha'} h'(r+\eta\varepsilon-\xi\delta) - D_{s-}^{1-\alpha'} h'(r)| \, dr \, ds \, d\xi \, d\eta.$$

Denote $\Delta = \xi\delta - \eta\varepsilon$ and

$$f_\Delta(t) := \int_0^t \int_0^s |D_{0+}^{\alpha'} \psi(r,s)| |[D_{s-}^{1-\alpha'} h'(r-\Delta) - D_{s-}^{1-\alpha'} h'(r)]| \, dr \, ds. \tag{3.9}$$

Then we may write

$$I_{\varepsilon,\delta} \le \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f_\Delta(t) \, d\xi \, d\eta. \tag{3.10}$$

Hence, in order to prove (3.8) it suffices to prove

$$f_\Delta(t) \le C \sup_{s\in[0,T]} \|\psi(\cdot,s)\|_\alpha |\Delta|^{2H+\gamma-1}. \tag{3.11}$$

By (2.2), we have

$$\begin{aligned} |D_{0+}^{\alpha'} \psi(r,s)| &= \frac{1}{\Gamma(1-\alpha')} \left| \frac{\psi(r,s)}{r^{\alpha'}} + \alpha' \int_0^r \frac{\psi(r,s) - \psi(u,s)}{(r-u)^{\alpha'+1}} \, du \right| \\ &\le C \sup_{s\in[0,T]} \|\psi(\cdot,s)\|_\alpha. \end{aligned} \tag{3.12}$$

Therefore,

$$f_\Delta(t) \le C \sup_{s\in[0,T]} \|\psi(\cdot,s)\|_\alpha (F_\Delta^1 + F_\Delta^2), \tag{3.13}$$

where

$$F_\Delta^1 = \int_0^t \int_0^s \frac{|h'(r-\Delta) - h'(r)|}{(s-r)^{1-\alpha'}}\,dr\,ds,$$

$$F_\Delta^2 = \int_0^t \int_0^s \int_r^s \frac{|h'(r-\Delta) - h'(u-\Delta) - h'(r) + h'(u)|}{(u-r)^{2-\alpha'}}\,du\,dr\,ds.$$

As in the proof of Lemma 3.2, we consider the two cases separately: $\Delta \le 0$ and $\Delta > 0$.

Case (i): If $\Delta \le 0$, we can write

$$\begin{aligned}\left|\frac{h'(r-\Delta) - h'(r)}{(s-r)^{1-\alpha'}}\right| &\le C(s-r)^{\alpha'-1}|\Delta| \int_0^1 (r-\xi\Delta)^{2H-2}\,d\xi \\ &\le C(s-r)^{\alpha'-1} r^{-\gamma} |\Delta|^{2H+\gamma-1},\end{aligned}$$

which implies

$$F_\Delta^1 \le C|\Delta|^{2H+\gamma-1}. \tag{3.14}$$

For $0 < r < u$, we have

$$\begin{aligned}&|h'(r-\Delta) - h'(u-\Delta) - h'(r) + h'(u)| \\ &\quad = C|\Delta| \int_0^1 \int_0^1 (r - \xi\Delta + \theta(u-r))^{2H-3}\,d\theta\,d\xi(r-u) \\ &\quad \le C r^{2H-1-\beta_1-\beta_2} (u-r)^{\beta_1} |\Delta|^{\beta_2}\end{aligned}$$

for any $\beta_1, \beta_2 > 0$ such that $\beta_1 + \beta_2 < 2H$. If $\alpha' + \beta_1 > 1$, we obtain

$$\begin{aligned}&\int_r^s \frac{|h'(r-\Delta) - h'(u-\Delta) - h'(r) + h(u)|}{(u-r)^{2-\alpha'}}\,du \\ &\qquad \le C r^{2H-1-\beta_1-\beta_2} (s-r)^{\alpha'+\beta_1-1} |\Delta|^{\beta_2},\end{aligned}$$

which implies, taking $\beta_2 = 2H + \gamma - 1$,

$$F_\Delta^2 \le C|\Delta|^{2H+\gamma-1}. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we get (3.11).

Case (ii): Now let $\Delta > 0$. We assume that $\Delta < t$ (the case $t \le \Delta$ is simpler and omitted). Let us first consider the term F_Δ^1. Define the sets

$$D_{11} = \{0 < r < s < \Delta\}, \qquad D_{12} = \{0 < r < \Delta < s < t\},$$
$$D_{13} = \{\Delta < r < s < t\}.$$

Then

$$F_\Delta^1 = F_\Delta^{11} + F_\Delta^{12} + F_\Delta^{13},$$

where

$$F_\Delta^{1i} = \int_{D_{1i}} \frac{|h'(r-\Delta) - h'(r)|}{(s-r)^{1-\alpha'}}\,dr\,ds, \qquad i = 1, 2, 3.$$

It is easy to see that

$$F_\Delta^{11} \le C\int_0^\Delta \int_0^s [(\Delta-r)^{2H-1}+r^{2H-1}](s-r)^{\alpha'-1}\,dr\,ds \le C\Delta^{2H+\alpha'} \tag{3.16}$$

and

$$F_\Delta^{12} \le C\int_\Delta^t \int_0^\Delta [(\Delta-r)^{2H-1}+r^{2H-1}](s-r)^{\alpha'-1}\,dr\,ds \le C\Delta^{2H}. \tag{3.17}$$

As for F_Δ^{13}, we have

$$F_\Delta^{13} = \int_\Delta^t \int_\Delta^s \frac{|h'(r-\Delta)-h'(r)|}{(s-r)^{1-\alpha}}\,dr\,ds = \int_0^{t-\Delta}\int_0^u \frac{|h'(v)-h'(v+\Delta)|}{(u-v)^{1-\alpha'}}\,dv\,du.$$

Using the estimate

$$|h'(v)-h'(v+\Delta)| \le Cv^{2H-\beta-1}\Delta^\beta$$

for all $0<\beta<2H$, we obtain

$$F_\Delta^{13} \le C\Delta^\beta. \tag{3.18}$$

Thus, (3.16)–(3.18) yield

$$F_\Delta^1 \le C\Delta^\beta \qquad \text{for all } 0<\beta<2H. \tag{3.19}$$

Now we study the second term F_Δ^2. Denote

$$D_{21} = \{0<r<u<s<\Delta<t\}, \qquad D_{22} = \{0<r<u<\Delta<s<t\},$$
$$D_{23} = \{0<r<\Delta<u<s<t\}, \qquad D_{24} = \{0<\Delta<r<u<s<t\}.$$

Then

$$F_\Delta^2 = F_\Delta^{21}+F_\Delta^{22}+F_\Delta^{23}+F_\Delta^{24},$$

where for $i=1,2,3,4$,

$$F_\Delta^{2i} = \int_{D_{2i}} \frac{|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u)|}{(u-r)^{2-\alpha'}}\,du\,dr\,ds.$$

Consider first the term F_Δ^{21}. We can write

$$\begin{aligned}\frac{1}{2H}|h'(r-\Delta)-h'(u-\Delta)| &= |(\Delta-u)^{2H-1}-(\Delta-r)^{2H-1}| \\ &\le C(u-r)\int_0^1 (\Delta-u+\theta(u-r))^{2H-2}\,d\theta \\ &\le C(u-r)^{1-\beta}(\Delta-u)^{2H+\beta-2},\end{aligned}$$

where $1-2H<\beta<\alpha'$. Similarly, we have

$$|h'(r)-h'(u)| \le Cr^{2H+\beta-2}(u-r)^{1-\beta}.$$

As a consequence,

$$
\begin{aligned}
F_\Delta^{21} &\le C \int_0^\Delta \int_0^s \int_r^s (u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,du\,dr\,ds \\
&\le C \int_0^\Delta \int_0^\Delta \int_r^\Delta (u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,du\,dr\,ds \qquad (3.20)\\
&\le C\Delta^{2H+\alpha'}.
\end{aligned}
$$

In a similar way we can prove that

$$
\begin{aligned}
F_\Delta^{22} &\le C \int_\Delta^t \int_0^\Delta \int_r^\Delta (u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,du\,dr\,ds \qquad (3.21)\\
&\le C\Delta^{2H+\alpha'-1}.
\end{aligned}
$$

For F_Δ^{23}, notice that when $r<\Delta<u$,

$$
\begin{aligned}
&|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u)| \\
&\qquad = (\Delta-r)^{2H-1}+(u-\Delta)^{2H-1}+r^{2H-1}+u^{2H-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(u-r)^{\alpha'-2} &= (u-\Delta+\Delta-r)^{\alpha'-2} \\
&\le (u-\Delta)^{-\beta}(\Delta-r)^{\alpha'+\beta-2} \wedge (u-\Delta)^{-\beta-2H+1}(\Delta-r)^{2H+\alpha'+\beta-3},
\end{aligned}
$$

where we can take any $\beta\in(0,1)$ satisfying $2H+\beta+\alpha'>2$. Then,

$$
\begin{aligned}
F_\Delta^{23} &\le C \int_{D_{23}} [(\Delta-r)^{2H-1}+(u-\Delta)^{2H-1}+r^{2H-1}+u^{2H-1}] \\
&\qquad \times (u-r)^{\alpha'-2}\,du\,dr\,ds \\
&\le C \int_{D_{23}} [(\Delta-r)^{2H+\alpha'+\beta-3}(u-\Delta)^{-\beta} \\
&\qquad + r^{2H-1}(u-\Delta)^{-\beta}(\Delta-r)^{\alpha'+\beta-2}]\,du\,dr\,ds \\
&\le C\Delta^{2H+\alpha'+\beta-2}.
\end{aligned}
$$

Taking $\beta=1+\gamma-\alpha'$, we obtain

$$
|F_\Delta^{23}| \le C\Delta^{2H+\alpha'-1}. \qquad (3.22)
$$

Finally we consider the last term F_Δ^{24}. Making the substitutions $x=r-\Delta$, $y=u-\Delta$ we can write

$$
\begin{aligned}
F_\Delta^{24} &= \int_{D_{24}} \frac{|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u)|}{(u-r)^{2-\alpha'}}\,du\,dr\,ds \\
&= \int_\Delta^t \int_0^{s-\Delta} \int_x^{s-\Delta} \frac{|h'(x)-h'(y)-h'(x+\Delta)+h'(y+\Delta)|}{(y-x)^{2-\alpha'}}\,dy\,dx\,ds.
\end{aligned}
$$

Note that for $0 < x < y$ and $\Delta > 0$,

$$\begin{aligned}
&|h'(x) - h'(y) - h'(x+\Delta) + h'(y+\Delta)| \\
&\quad = x^{2H-1} - y^{2H-1} - (x+\Delta)^{2H-1} + (y+\Delta)^{2H-1} \\
&\quad = C\int_0^1\int_0^1 (x + \theta(y-x) + \tilde{\theta}\Delta)^{2H-3}\,d\theta\,d\tilde{\theta} \\
&\quad \le Cx^{2H+\beta_1+\beta_2-3}(y-x)^{1-\beta_1}\Delta^{1-\beta_2},
\end{aligned}$$

where

$$0 < \beta_1, \beta_2 < 1, \qquad 2H + \beta_1 + \beta_2 > 2 \quad \text{and} \quad \beta_1 < \alpha'.$$

Taking $\beta_2 = 2 - 2H - \gamma$ we get

$$F_\Delta^{24} \le C\Delta^{2H+\gamma-1}. \tag{3.23}$$

From (3.20)–(3.23), we see that

$$F_\Delta^{2} \le C\Delta^{2H+\gamma-1}. \tag{3.24}$$

This completes the proof of the lemma. □

THEOREM 3.4. *Suppose that $\phi \in C^\alpha([0,T])$ with $\gamma\alpha > 1 - 2H$ on $[0,T]$. Then, the nonlinear stochastic integral $\int_0^t W(ds, \phi_s)$ exists and*

$$\begin{aligned}
&E\biggl(\int_0^t W(ds,\phi_s)\biggr)^2 \\
&\qquad = 2H\int_0^t \theta^{2H-1}Q(\phi_\theta,\phi_\theta)\,d\theta \\
&\qquad\quad + 2H(2H-1)\int_0^t\int_0^\theta r^{2H-2}(Q(\phi_\theta,\phi_{\theta-r}) - Q(\phi_\theta,\phi_\theta))\,dr\,d\theta.
\end{aligned} \tag{3.25}$$

Furthermore, for any $\frac{1-2H}{\gamma} < \alpha' < \alpha$, we have

$$\begin{aligned}
&\sup_{0\le t\le T} E\biggl(\biggl|\int_0^t \dot{W}^\varepsilon(s,\phi_s)\,ds - \int_0^t W(ds,\phi_s)\biggr|^2\biggr) \\
&\qquad \le C(1 + \|\phi\|_\infty)^M (1 + \|\phi\|_\alpha^\gamma)\varepsilon^{2H+\gamma\alpha'-1},
\end{aligned} \tag{3.26}$$

where the constant C depends on H, T, γ, α, α' and the constants C_0 and C_1 appearing in (Q1) and (Q2).

PROOF. We can write (3.2) as

$$\begin{aligned}
E(I_\varepsilon(\phi)I_\delta(\phi)) &= \int_0^t\int_0^\theta (Q(\phi_\theta,\phi_{\theta-r}) - Q(\phi_\theta,\phi_\theta))V_{\varepsilon,\delta}^{2H}(r)\,dr\,d\theta \\
&\quad + \int_0^t\int_0^\theta Q(\phi_\theta,\phi_\theta)V_{\varepsilon,\delta}^{2H}(r)\,dr\,d\theta.
\end{aligned} \tag{3.27}$$

Due to the local boundedness of Q [see (Q1)] and applying Lemma 3.2 to $\psi(\theta)=Q(\phi_\theta,\phi_\theta)$, we see that the second integral converges to

$$\lim_{\varepsilon,\delta\to 0}\int_0^t\int_0^\theta Q(\phi_\theta,\phi_\theta)V_{\varepsilon,\delta}^{2H}(r)\,dr\,d\theta=2H\int_0^t Q(\phi_\theta,\phi_\theta)\theta^{2H-1}\,d\theta.$$

On the other hand, using the local Hölder continuity of Q [see (Q2)] and applying Lemma 3.3, to $\psi(r,\theta)=Q(\phi_\theta,\phi_{\theta-r})-Q(\phi_\theta,\phi_\theta)$, we see that the first integral converges to

$$\begin{aligned}&\lim_{\varepsilon,\delta\to 0}\int_0^t\int_0^\theta (Q(\phi_\theta,\phi_{\theta-r})-Q(\phi_\theta,\phi_\theta))V_{\varepsilon,\delta}^{2H}(r)\,dr\,d\theta\\&\qquad=2H(2H-1)\int_0^t\int_0^\theta (Q(\phi_\theta,\phi_{\theta-r})-Q(\phi_\theta,\phi_\theta))r^{2H-2}\,dr\,d\theta.\end{aligned}$$

This implies that $\{I_{\varepsilon_n}(\phi),n\geq 1\}$ is a Cauchy sequence in $L^2(\Omega)$ for any sequence $\varepsilon_n\downarrow 0$. As a consequence, $\lim_{\varepsilon\to 0}I_\varepsilon(\phi)$ exists in $L^2(\Omega)$ and is denoted by $I(\phi):=\int_0^t W(ds,\phi_s)$. Letting $\varepsilon,\delta\to 0$ in (3.27), we obtain (3.25).

From (3.27), Lemma 3.2 and Lemma 3.3, we have for any $\alpha'<\alpha$,

$$\begin{aligned}&|E(I_\varepsilon(\phi)I_\delta(\phi))-E(I^2(\phi))|\\&\qquad\leq C(1+\|\phi\|_\infty)^M(1+\|\phi\|_\alpha^\gamma)(\varepsilon+\delta)^{2H+\gamma\alpha'-1}.\end{aligned}\tag{3.28}$$

In equation (3.28), let $\delta\to 0$ and notice that $I_\delta(\phi)\to I(\phi)$ in $L^2(\Omega)$. Then

$$|E(I_\varepsilon(\phi)I(\phi))-E(I^2(\phi))|\leq C(1+\|\phi\|_\infty)^M(1+\|\phi\|_\alpha^\gamma)\varepsilon^{2H+\gamma\alpha'-1}.$$

On the other hand, if we let $\varepsilon=\delta$ in (3.28), we obtain

$$|EI_\varepsilon^2(\phi)-E(I^2(\phi))|\leq C(1+\|\phi\|_\infty)^M(1+\|\phi\|_\alpha^\gamma)\varepsilon^{2H+\gamma\alpha'-1}.$$

Thus, we have

$$E|I_\varepsilon(\phi)-I(\phi)|^2=[E(I_\varepsilon^2(\phi))-E(I^2(\phi))]-2[E(I_\varepsilon(\phi)I(\phi))-E(I^2(\phi))].$$

Applying the triangular inequality, we obtain (3.26). $\square$

The following proposition can be proved in the same way as (3.25).

PROPOSITION 3.5. *Suppose $\phi,\psi\in C^\alpha([0,T])$ with $\alpha\gamma>1-2H$. Then*

$$\begin{aligned}&E\left(\int_0^t W(dr,\phi_r)\int_0^t W(dr,\psi_r)\right)\\&\qquad=2H\int_0^t\theta^{2H-1}Q(\phi_\theta,\psi_\theta)\,d\theta\end{aligned}\tag{3.29}$$

$$+ H(2H-1)\int_0^t \int_0^\theta r^{2H-2}(Q(\phi_\theta, \psi_{\theta-r}) - Q(\phi_\theta, \psi_\theta))\, dr\, d\theta$$

$$+ H(2H-1)\int_0^t \int_0^\theta r^{2H-2}(Q(\phi_{\theta-r}, \psi_\theta) - Q(\phi_\theta, \psi_\theta))\, dr\, d\theta.$$

The following proposition provides the Hölder continuity of the indefinite integral.

PROPOSITION 3.6. *Suppose* $\phi \in C^\alpha([0,T])$ *with* $\alpha\gamma > 1 - 2H$. *Then for all* $0 \le s < t \le T$,

$$E\biggl(\int_0^t W(dr, \phi_r) - \int_0^s W(dr, \phi_r)\biggr)^2 \le C(1 + \|\phi\|_\infty)^M (t-s)^{2H}, \tag{3.30}$$

where the constant C *depends on* H, T, γ, α *and the constants* C_0 *and* C_1 *appearing in* (Q1) *and* (Q2). *As a consequence, the process* $X_t = \int_0^t W(dr, \phi_r)$ *is almost surely* $(H - \delta)$*-Hölder continuous for any* $\delta > 0$.

PROOF. We shall first show that

$$E\biggl(\int_0^t \dot W^\varepsilon(r, \phi_r)\, dr - \int_0^s \dot W^\varepsilon(r, \phi_r)\, dr\biggr)^2 \le C(1 + \|\phi\|_\infty)^M (t-s)^{2H}. \tag{3.31}$$

We can write

$$\begin{aligned}
E\biggl(&\int_0^t W^\varepsilon(dr, \phi_r) - \int_0^s W^\varepsilon(dr, \phi_r)\biggr)^2 \\
&= E\biggl(\int_s^t W^\varepsilon(dr, \phi_r)\biggr)^2 \\
&= \frac{1}{4\varepsilon^2}\int_s^t \int_s^t E[(W(\theta+\varepsilon, \phi_\theta) - W(\theta-\varepsilon, \phi_\theta)) \\
&\qquad\qquad \times (W(\eta+\varepsilon, \phi_\eta) - W(\eta-\varepsilon, \phi_\eta))]\, d\theta\, d\eta \\
&= \frac{1}{8\varepsilon^2}\int_s^t \int_s^t Q(\phi_\theta, \phi_\eta) \\
&\qquad\qquad \times [|\eta-\theta|^{2H} - |\eta-\theta-2\varepsilon|^{2H} - |\eta-\theta+2\varepsilon|^{2H}]\, d\theta\, d\eta \\
&= \frac{1}{8\varepsilon^2}\int_0^{t-s} \int_0^{t-s} Q(\phi_{s+\theta}, \phi_{s+\eta}) \\
&\qquad\qquad \times [|\eta-\theta|^{2H} - |\eta-\theta-2\varepsilon|^{2H} - |\eta-\theta+2\varepsilon|^{2H}]\, d\theta\, d\eta \\
&= \frac{1}{4\varepsilon^2}\int_0^{t-s} \int_0^\theta Q(\phi_{s+\theta}, \phi_{s+\theta-r})[2r^{2H} - |r+2\varepsilon|^{2H} - |r-2\varepsilon|^{2H}]\, dr\, d\theta.
\end{aligned}$$

The inequality (3.31) follows from the assumption (Q1) and the inequality (A.2) obtained in the Appendix. Finally, the inequality (3.30) follows from (3.31), Proposition 3.4 and the Fatou's lemma. $\square$

4. Feynman–Kac integral. In this section, we show that the random field $u(t,x)$ given by (1.2) is well defined and study its Hölder continuity. Since the Brownian motion B_t has Hölder continuous trajectories of order δ for any $\delta \in (0, \frac{1}{2})$, by Lemma 3.4 the nonlinear stochastic integral $\int_0^t W(ds, B_{t-s}^x)$ can be defined for any $H > \frac{1}{2} - \frac{\gamma}{4}$. The following theorem shows that it is exponentially integrable and hence $u(t,x)$ is well defined.

Set $\|B\|_{\infty,T} = \sup_{0 \le s \le T} |B_s|$ and $\|B\|_{\delta,T} = \sup_{0 \le s < t \le T} \frac{|B_t - B_s|}{|t-s|^\delta}$ for $\delta \in (0, \frac{1}{2})$.

THEOREM 4.1. *Let $H > \frac{1}{2} - \frac{\gamma}{4}$ and let u_0 be bounded. For any $t \in [0,T]$ and $x \in \mathbb{R}^d$, the random variable $\int_0^t W(ds, B_{t-s}^x)$ is exponentially integrable and the random field $u(t,x)$ given by (1.2) is in $L^p(\Omega)$ for any $p \ge 1$.*

PROOF. Suppose first that $p = 1$. By (3.30) with $s = 0$ and the Fernique's theorem we have

$$\begin{aligned} E^W |u(t,x)| &\le \|u_0\|_\infty E^B E^W \left[\exp \int_0^t W(ds, B_{t-s}^x) \right] \\ &\le \|u_0\|_\infty E^B [e^{Ct^{2H}(1+\|B\|_{\infty,T})^M}] < \infty. \end{aligned}$$

The L^p integrability of $u(t,x)$ follows from Jensen's inequality

$$\begin{aligned} E^W |u(t,x)|^p &\le \|u_0\|_\infty E^B E^W \exp\left(p \int_0^t W(dr, B_{t-r}^x) \right) \\ &\le \|u_0\|_\infty E^B [\exp(Cp(1 + \|B\|_{\infty,T})^M T^{2H})] < \infty. \end{aligned} \tag{4.1}$$

$\square$

To show the Hölder continuity of $u(\cdot, x)$, we need the following lemma.

LEMMA 4.2. *Assume that u_0 is Lipschitz continuous. Then for $0 \le s < t \le T$ and for any $\alpha < 2H - 1 + \frac{1}{2}\gamma$,*

$$E^W \left| \int_0^s W(dr, B_{t-r}^x) - \int_0^s W(dr, B_{s-r}^x) \right|^2 \le C(1 + \|B\|_{\infty,T})^M \|B\|_{\delta,T}^\gamma (t-s)^\alpha,$$

where the constant C depends on H, T, γ and the constant C_1 appearing in (Q2).

PROOF. Suppose $\delta \in (0, \frac{1}{2})$. For $0 \le u < v < s \le T$, denote

$$\begin{aligned} \Delta Q(s,t,u,v) := Q(B_{t-u}^x, B_{t-v}^x) - Q(B_{t-u}^x, B_{t-u}^x) \\ - Q(B_{t-u}^x, B_{s-v}^x) + Q(B_{t-u}^x, B_{s-u}^x). \end{aligned}$$

Note that (Q2) implies

$$|\Delta Q(s,t,u,v)| \le 2C_1(1+\|B\|_{\infty,T})^M \|B\|_{\delta,T}^{\gamma}(t-s)^{\gamma\delta}$$

and

$$|\Delta Q(s,t,u,v)| \le 2C_1(1+\|B\|_{\infty,T})^M \|B\|_{\delta,T}^{\gamma}|u-v|^{\gamma\delta},$$

which imply that for any $\beta \in (0,1)$,

$$|\Delta Q(s,t,u,v)| \le 2C_1(1+\|B\|_{\infty,T})^M \|B\|_{\delta,T}^{\gamma}(t-s)^{\beta\gamma\delta}|u-v|^{(1-\beta)\gamma\delta}.$$

Applying (3.29) and using $Q(x,y) = Q(y,x)$, we get

$$\begin{aligned}
E^W & \left|\int_0^s W(dr, B_{t-r}^x) - \int_0^s W(dr, B_{s-r}^x)\right|^2 \\
&= 2H(2H-1) \\
&\quad \times \int_0^s \int_0^\theta r^{2H-2}[\Delta Q(s,t,\theta,\theta-r) + \Delta Q(t,s,\theta,\theta-r)]\,dr\,d\theta \\
&\quad + 2H \int_0^s \theta^{2H-1} \\
&\qquad\qquad \times [Q(B_{t-\theta}^x, B_{t-\theta}^x) - 2Q(B_{t-\theta}^x, B_{s-\theta}^x) + Q(B_{s-\theta}^x, B_{s-\theta}^x)]\,d\theta \\
&\le C(1+\|B\|_{\infty,T})^M \|B\|_{\delta,T}^{\gamma}(t-s)^{\beta\gamma\delta}
\end{aligned}$$

for any β such that $(1-\beta)\gamma\delta > 1-2H$, that is, $\beta\gamma\delta < 2H-1+\gamma\delta$. Taking β and δ such that $\beta\gamma\delta = \alpha$, we get the lemma. □

THEOREM 4.3. *Suppose u_0 is Lipschitz continuous and bounded. Then for each $x \in \mathbb{R}^d$, $u(\cdot,x) \in C^{H_1}([0,T])$ for any $H_1 \in (0, H-\frac{1}{2}+\frac{1}{4}\gamma)$.*

PROOF. For $0 \le s < t \le T$, from the Minkowski's inequality it follows that

$$\begin{aligned}
E^W & [|u(t,x) - u(s,x)|^p] \\
&\le [E^B(E^W|u_0(B_t^x)e^{\int_0^t W(dr,B_{t-r}^x)} - u_0(B_s^x)e^{\int_0^s W(dr,B_{s-r}^x)}|^p)^{1/p}]^p \\
&\le C\|u_0\|_\infty [E^B(E^W|e^{\int_0^t W(dr,B_{t-r}^x)} - e^{\int_0^s W(dr,B_{s-r}^x)}|^p)^{1/p}]^p \\
&\quad + C[E^B(E^W|(u_0(B_t^x) - u_0(B_s^x))e^{\int_0^s W(dr,B_{s-r}^x)}|^p)^{1/p}]^p.
\end{aligned} \tag{4.2}$$

Since u_0 is Lipschitz continuous, using (4.1) and Hölder's inequality, we have

$$[E^B(E^W(|u_0(B_t^x) - u_0(B_s^x)|e^{\int_0^s W(dr,B_{s-r}^x)})^p)^{1/p}]^p \le C(t-s)^{p/2}. \tag{4.3}$$

For the first term in (4.2), using the formula that $|e^a - e^b| \le (e^a + e^b)|a-b|$ for $a, b \in \mathbb{R}$ and Hölder's inequality we get

$$
\begin{aligned}
&E^W\left[\left|\exp\int_0^t W(dr, B^x_{t-r}) - \exp\int_0^s W(dr, B^x_{s-r})\right|^p\right] \\
&\quad\le \left[E^W\left(\exp\int_0^t W(dr, B^x_{t-r}) + \exp\int_0^s W(dr, B^x_{s-r})\right)^{2p}\right]^{1/2} \qquad (4.4)\\
&\qquad \times \left[E^W\left|\int_0^t W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r})\right|^{2p}\right]^{1/2}.
\end{aligned}
$$

Applying Lemma 3.6 and Lemma 4.2, we obtain

$$
\begin{aligned}
&E^W\left|\int_0^t W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r})\right|^2 \\
&\quad\le 2E^W\left|\int_s^t W(dr, B^x_{t-r})\right|^2 \\
&\qquad + 2E^W\left|\int_0^s W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r})\right|^2 \qquad (4.5)\\
&\quad\le C(1 + \|B\|_{\infty,T})^M \|B\|^\gamma_{\delta,T}(t-s)^{2H_1}.
\end{aligned}
$$

Noting that conditional to B, $\int_0^t W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r})$ is Gaussian, and using (4.4), (4.5) and (4.1) we get

$$
\begin{aligned}
&\left[E^B\left(E^W\left|\exp\int_0^t W(dr, B^x_{t-r}) - \exp\int_0^s W(dr, B^x_{s-r})\right|^p\right)^{1/p}\right]^p \\
&\quad\le C\left[E^B\left(E^W\left|\int_0^t W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r})\right|^2\right)^{1/2}\right]^p \qquad (4.6)\\
&\quad\le C(t-s)^{pH_1}.
\end{aligned}
$$

From (4.2), (4.3) and (4.6), we can see that for any $p \ge 1$,

$$
E^W[|u(t,x) - u(s,x)|^p] \le C(t-s)^{pH_1}. \qquad (4.7)
$$

Now Kolmogorov's continuity criterion implies the theorem. □

5. Validation of the Feynman–Kac formula.

In the last section, we have proved that $u(t,x)$ given by (1.2) is well defined. In this section, we shall show that $u(t,x)$ is a weak solution to equation (1.1).

To give the exact meaning about what we mean by a weak solution, we follow the idea of [3] and [4]. First, we need a definition of the Stratonovich integral.

DEFINITION 5.1. Given a random field $v = \{v(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ such that $\int_0^t \int_{\mathbb{R}^d} |v(s,x)|\, dx\, ds < \infty$ a.s. for all $t > 0$, the Stratonovich integral

$$\int_0^t \int_{\mathbb{R}^d} v(s,x) W(ds,x)\, dx$$

is defined as the following limit in probability if it exists

$$\lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^d} v(s,x) \dot{W}^\varepsilon(s,x)\, ds\, dx,$$

where $W^\varepsilon(t,x)$ is introduced in (3.1).

The precise meaning of the weak solution to equation (1.1) is given below.

DEFINITION 5.2. A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a weak solution to equation (1.1) if for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (u(t,x) - u_0(x))\varphi(x)\, dx &= \int_0^t \int_{\mathbb{R}^d} u(s,x) \Delta\varphi(x)\, dx\, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x) W(ds,x)\, dx \end{aligned} \tag{5.1}$$

almost surely, for all $t \geq 0$, where the last term is a Stratonovich stochastic integral in the sense of Definition 5.1.

The following theorem justifies the Feynman–Kac formula (1.2).

THEOREM 5.3. *Suppose $H > \frac{1}{2} - \frac{1}{4}\gamma$ and u_0 is a bounded measurable function. Let $u(t,x)$ be the random field defined in (1.2). Then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, $u(t,x)\varphi(x)$ is Stratonovich integrable and $u(t,x)$ is a weak solution to equation (1.1) in the sense of Definition 5.2.*

PROOF. We prove this theorem by a limit argument. We divide the proof into three steps.

Step 1. Let $u^\varepsilon(t,x)$ be the unique solution to the following equation:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\Delta u^\varepsilon + u^\varepsilon \dfrac{\partial W^\varepsilon}{\partial t}(t,x), & t > 0, x \in \mathbb{R}^d, \\ u^\varepsilon(0,x) = u_0(x). \end{cases} \tag{5.2}$$

Since $W^\varepsilon(t,x)$ is differentiable, the classical Feynman–Kac formula holds for the solution to this equation, that is,

$$u^\varepsilon(t,x) := E^B[u_0(B_t^x) e^{\int_0^t \dot{W}^\varepsilon(s, B_{t-s}^x)\, ds}].$$

The fact that $u^\varepsilon(t,x)$ is well defined follows from (3.31) and Fernique's theorem. In fact, we have (cf. the argument in the proof of Lemma 4.1)

$$
\begin{aligned}
E^W|u^\varepsilon(t,x)|^p &\le \|u_0\|_\infty E^B E^W \exp\left(p\int_0^t \dot W^\varepsilon(r,B^x_{t-r})\,dr\right)\\
&\le \|u_0\|_\infty E^B[\exp(Cp(1+\|B\|_{\infty,T})^M t^{2H})] < \infty.
\end{aligned} \tag{5.3}
$$

Introduce the following notations

$$
\begin{aligned}
g^\varepsilon_{s,x}(r,z) &:= \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z),\\
g^B_{s,x}(r,z) &:= \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,B^x_{s-r}]}(z),\\
g^{\varepsilon,B}_{s,x}(r,z) &:= \int_0^s \frac{1}{\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)\mathbf{1}_{[0,B^x_{s-\theta}]}(z)\,d\theta.
\end{aligned}
$$

From the results of Section 3, we see that $g^\varepsilon_{s,x}$, $g^B_{s,x}$, $g^{\varepsilon,B}_{s,x} \in \mathcal{H}$ ($\mathcal{H}$ is introduced in Section 2), and we can write

$$
\dot W^\varepsilon(s,x) = W\left(\frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\right) = W(g^\varepsilon_{s,x}),
$$

$$
\int_0^s W(d\theta, B^x_{s-\theta}) = W(g^B_{s,x}), \qquad \int_0^s \dot W^\varepsilon(\theta, B^x_{s-\theta})\,d\theta = W(g^{\varepsilon,B}_{s,x}).
$$

Set

$$
\widetilde{u}^\varepsilon(s,x) := u^\varepsilon(s,x) - u(s,x).
$$

Step 2. We prove the following claim:

$u^\varepsilon(s,x)\to u(s,x)$ in $\mathbb{D}^{1,2}$ as $\varepsilon\downarrow 0$, uniformly on any compact subset of $[0,T]\times\mathbb{R}^d$, that is, for any compact $K\subseteq\mathbb{R}^d$

$$
\sup_{s\in[0,T],x\in K} E^W[|\widetilde{u}^\varepsilon(s,x)|^2 + \|D\widetilde{u}^\varepsilon(s,x)\|^2_{\mathcal{H}}] \to 0 \qquad \text{as } \varepsilon\downarrow 0. \tag{5.4}
$$

Since u_0 is bounded, without loss of generality, we may assume $u_0\equiv 1$. Let B^1 and B^2 be two independent Brownian motions, both independent of W. Using the inequality $|e^a-e^b|\le(e^a+e^b)|a-b|$, Hölder inequality and the fact that $W(g^{\varepsilon,B}_{t,x})$ and $W(g^B_{t,x})$ are Gaussian conditioning to B, we have

$$
\begin{aligned}
&E^W(u^\varepsilon(t,x)-u(t,x))^2\\
&\quad = E^W[E^B(e^{W(g^{\varepsilon,B}_{t,x})} - e^{W(g^B_{t,x})})]^2\\
&\quad \le E^B E^W|e^{W(g^{\varepsilon,B}_{t,x})} - e^{W(g^B_{t,x})}|^2\\
&\quad \le E^B[E^W(e^{W(g^{\varepsilon,B}_{t,x})} + e^{W(g^B_{t,x})})^4]^{1/2}[E^W|W(g^{\varepsilon,B}_{t,x}) - W(g^B_{t,x})|^4]^{1/2}\\
&\quad \le C[E^B E^W(e^{4W(g^{\varepsilon,B}_{t,x})} + e^{4W(g^B_{t,x})})]^{1/2} E^B E^W|W(g^{\varepsilon,B}_{t,x}) - W(g^B_{t,x})|^2.
\end{aligned}
$$

Note that (4.1) and (5.3) imply

$$E^B E^W (e^{pW(g^{\varepsilon,B}_{t,x})} + e^{pW(g^B_{t,x})}) < \infty \tag{5.5}$$

for any $p \geq 1$. On the other hand, applying Theorem 3.4, we have

$$\sup_{0\leq t\leq T, x\in K} E^B E^W |W(g^{\varepsilon,B}_{t,x}) - W(g^B_{t,x})|^2 \to 0 \qquad \text{as } \varepsilon \downarrow 0. \tag{5.6}$$

Then it follows that as $\varepsilon \downarrow 0$

$$\sup_{0\leq t\leq T, x\in K} E^W |\widetilde{u}^\varepsilon(t,x)|^2 = \sup_{0\leq t\leq T, x\in K} E^W (u^\varepsilon(t,x) - u(t,x))^2 \to 0.$$

For the Malliavin derivatives, we have

$$Du^\varepsilon(s,x) = E^B[\exp(W(g^{\varepsilon,B}_{s,x}))g^{\varepsilon,B}_{s,x}],$$
$$Du(s,x) = E^B[\exp(W(g^B_{s,x}))g^B_{s,x}].$$

Then

$$\begin{aligned}
&E^W \|Du^\varepsilon(s,x) - Du(s,x)\|^2_{\mathcal{H}} \\
&\quad = E^W \|E^B[(\exp(W(g^{\varepsilon,B}_{s,x}))g^{\varepsilon,B}_{s,x} - \exp(W(g^B_{s,x}))g^B_{s,x})]\|^2_{\mathcal{H}} \\
&\quad \leq 2E^W E^B[\exp(2W(g^{\varepsilon,B}_{s,x}))\|g^{\varepsilon,B}_{s,x} - g^B_{s,x}\|^2_{\mathcal{H}}] \\
&\qquad + 2E^W E^B[|\exp(W(g^{\varepsilon,B}_{s,x})) - \exp(W(g^B_{s,x}))|^2 \|g^B_{s,x}\|^2_{\mathcal{H}}].
\end{aligned}$$

Note that $\|g^{\varepsilon,B}_{t,x} - g^B_{t,x}\|^2_{\mathcal{H}} = E^W|W(g^{\varepsilon,B}_{t,x}) - W(g^B_{t,x})|^2$. Then it follows again from (5.5) and (5.6) that as $\varepsilon \downarrow 0$

$$\sup_{0\leq t\leq T, x\in K} E^W \|Du^\varepsilon(s,x) - Du(s,x)\|^2_{\mathcal{H}} \to 0.$$

Step 3. From equation (5.2) and (5.4), it follows that $\int_0^t \int_{\mathbb{R}^d} u^\varepsilon(s,x)\varphi(x) \times \dot{W}^\varepsilon(s,x)\,ds\,dx$ converges in L^2 to some random variable as $\varepsilon \downarrow 0$. Hence, if

$$V_\varepsilon := \int_0^t \int_{\mathbb{R}^d} (u^\varepsilon(s,x) - u(s,x))\varphi(x)\dot{W}^\varepsilon(s,x)\,ds\,dx \tag{5.7}$$

converges to zero in L^2, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}^d} u(s,x)\varphi(x)\dot{W}^\varepsilon(s,x)\,ds\,dx \\
&\qquad = \lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}^d} u^\varepsilon(s,x)\varphi(x)\dot{W}^\varepsilon(s,x)\,ds\,dx,
\end{aligned}$$

that is, $u(s,x)\varphi(x)$ is Stratonovich integrable and $u(s,x)$ is a weak solution to equation (1.1). Thus, it remains to show that V_ε converges to zero in L^2.

In order to show the convergence to zero of (5.7) in L^2, first we write $\widetilde{u}^\varepsilon(s,x)W(g^\varepsilon_{s,x})$ as the sum of a divergence integral and a trace term [see (2.1)]

$$\widetilde{u}^\varepsilon(s,x)W(g^\varepsilon_{s,x}) = \delta(\widetilde{u}^\varepsilon(s,x)g^\varepsilon_{s,x}) - \langle D\widetilde{u}^\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}}.$$

Then we have

$$\begin{aligned} V_\varepsilon &= \int_0^t \int_{\mathbb{R}^d} \widetilde{u}^\varepsilon(s,x)\varphi(x)W(g^\varepsilon_{s,x})\,ds\,dx \\ &= \int_0^t \int_{\mathbb{R}^d} (\delta(\widetilde{u}^\varepsilon(s,x)g^\varepsilon_{s,x}) - \langle D\widetilde{u}^\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}})\varphi(x)\,ds\,dx \\ &= \delta(\psi^\varepsilon) - \int_0^t \int_{\mathbb{R}^d} \langle D\widetilde{u}^\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}}\varphi(x)\,ds\,dx =: V^1_\varepsilon - V^2_\varepsilon, \end{aligned}$$

where

$$\psi^\varepsilon(r,z) = \int_0^t \int_{\mathbb{R}^d} \widetilde{u}^\varepsilon(s,x)g^\varepsilon_{s,x}(r,z)\varphi(x)\,ds\,dx.$$

For the term V^1_ε, using the estimates on L^2 norm of the Skorohod integral (see (1.47) in [7]), we obtain

$$E[|V^1_\varepsilon|^2] \le E[\|\psi^\varepsilon\|^2_{\mathcal{H}}] + E[\|D\psi^\varepsilon\|^2_{\mathcal{H}\otimes\mathcal{H}}]. \tag{5.8}$$

Denoting $\operatorname{supp}(\varphi)$ the support of φ, we have

$$\begin{aligned} &E[\|\psi^\varepsilon\|^2_{\mathcal{H}}] \\ &\quad = E\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widetilde{u}^\varepsilon(s_1,x_1)\widetilde{u}^\varepsilon(s_2,x_2) \\ &\qquad\qquad \times \langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\rangle_{\mathcal{H}}\varphi(x_1)\varphi(x_2)\,ds_1\,ds_2\,dx_1\,dx_2 \\ &\quad \le M_1 \int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\rangle_{\mathcal{H}}\varphi(x_1)\varphi(x_2)\,ds_1\,ds_2\,dx_1\,dx_2 \\ &\quad = M_1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} E^W[W^\varepsilon(t,x_1)W^\varepsilon(t,x_2)]\varphi(x_1)\varphi(x_2)\,dx_1\,dx_2, \end{aligned}$$

where $M_1 := \sup_{s\in[0,T],x\in\operatorname{supp}(\varphi)} E[|\widetilde{u}^\varepsilon(s,x)|^2]$. Note that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} E^W[W^\varepsilon(t,x_1)W^\varepsilon(t,x_2)]\varphi(x_1)\varphi(x_2)\,dx_1\,dx_2 \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} E^W[W(t,x_1)W(t,x_2)]\varphi(x_1)\varphi(x_2)\,dx_1\,dx_2 \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} t^{2H}Q(x_1,x_2)\varphi(x_1)\varphi(x_2)\,dx_1\,dx_2 < \infty. \end{aligned} \tag{5.9}$$

Thus by (5.4), we get $E[\|\psi^\varepsilon\|^2_{\mathcal{H}}] \to 0$ as $\varepsilon \downarrow 0$.

On the other hand, setting $M_2 := \sup_{s\in[0,T],x\in\mathrm{supp}(\varphi)} E[\|D\widetilde{u}^\varepsilon(s,x)\|_{\mathcal{H}}^2]$, we have

$$\begin{aligned}
&E[\|D\psi^\varepsilon\|_{\mathcal{H}\otimes\mathcal{H}}^2] \\
&\quad= E\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle D\widetilde{u}^\varepsilon(s_1,x_1)\otimes g^\varepsilon_{s_1,x_1}, D\widetilde{u}^\varepsilon(s_2,x_2)\otimes g^\varepsilon_{s_2,x_2}\rangle_{\mathcal{H}} \\
&\qquad\qquad \times \varphi(x_1)\varphi(x_2)\,ds_1\,ds_2\,dx_1\,dx_2 \\
&\quad= E\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle D\widetilde{u}^\varepsilon(s_1,x_1), D\widetilde{u}^\varepsilon(s_2,x_2)\rangle_{\mathcal{H}}\langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\rangle_{\mathcal{H}} \\
&\qquad\qquad \times \varphi(x_1)\varphi(x_2)\,ds_1\,ds_2\,dx_1\,dx_2 \\
&\quad\le M_2\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\rangle_{\mathcal{H}}\varphi(x_1)\varphi(x_2)\,ds_1\,ds_2\,dx_1\,dx_2.
\end{aligned}$$

Then (5.4) and (5.9) imply that $E[\|D\psi^\varepsilon\|_{\mathcal{H}\otimes\mathcal{H}}^2]$ converges to zero as $\varepsilon\downarrow 0$.

Finally, we deal with the trace term

$$\begin{aligned}
V_\varepsilon^2 = {}&\int_0^t\int_{\mathbb{R}^d}(\langle Du^\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}} - \langle Du(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}})\varphi(x)\,ds\,dx \\
=: {}& T_1^\varepsilon - T_2^\varepsilon,
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
T_1^\varepsilon &= \int_0^t\int_{\mathbb{R}^d}\langle Du^\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}}\varphi(x)\,ds\,dx, \\
T_2^\varepsilon &= \int_0^t\int_{\mathbb{R}^d}\langle Du(s,x), g^\varepsilon_{s,x}\rangle_{\mathcal{H}}\varphi(x)\,ds\,dx.
\end{aligned}$$

We will show that T_1^ε and T_2^ε converge to the same random variable as $\varepsilon\downarrow 0$.

We start with the term T_2^ε. Note that

$$\begin{aligned}
\langle g^B_{s,x}, g^\varepsilon_{s,x}\rangle &= \left\langle \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,B^x_{s-r}]}(z), \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\right\rangle_{\mathcal{H}} \\
&= \left\langle \mathbf{1}_{[0,s]}(r)Q(B^x_{s-r},x), \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\right\rangle_{\mathcal{H}}.
\end{aligned}$$

Since $Q(B^x_{s-\cdot},x)\in C^{1/2-\delta}([0,T])$ for any $0<\delta<\frac{1}{2}$, noticing that $H>\frac{1}{2}-\frac{\gamma}{4}$ and applying Lemma A.4 we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} T_2^\varepsilon &= E^B\int_0^t\int_{\mathbb{R}^d} u_0(B^x_s)\exp(W(g^B_{s,x}))\langle g^B_{s,x}, g^\varepsilon_{s,x}\rangle_{\mathcal{H}}\varphi(x)\,ds\,dx \\
&= E^B\int_0^t\int_{\mathbb{R}^d} u_0(B^x_s)\exp(W(g^B_{s,x}))\varphi(x)
\end{aligned}$$

$$\begin{aligned} &\times \biggl[Q(x,x)Hs^{2H-1} \\ &\quad + H(2H-1)\int_0^s (Q(B^x_{s-r},x) \\ &\quad - Q(x,x))r^{2H-2}\,dr\biggr]\,ds\,dx. \end{aligned} \tag{5.11}$$

On the other hand, for the term T_1^ε, note that

$$\begin{aligned} \langle g^{\varepsilon,B}_{s,x}, g^\varepsilon_{s,x}\rangle &= \biggl\langle \int_0^{2\varepsilon} \frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)\mathbf{1}_{[0,B^x_{s-\theta}]}(z)\,d\theta, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\biggr\rangle_{\mathcal{H}} \\ &= \biggl\langle \int_0^{2\varepsilon} \frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)Q(B^x_{s-\theta},x)\,d\theta, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\biggr\rangle_{\mathcal{H}}. \end{aligned}$$

Applying Lemma A.5, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} T_1^\varepsilon &= E^B\int_0^t\int_{\mathbb{R}^d} u_0(B^x_s)\exp(W(g^{\varepsilon,B}_{s,x}))\langle g^{\varepsilon,B}_{s,x}, g^\varepsilon_{s,x}\rangle_{\mathcal{H}}\varphi(x)\,ds\,dx \\ &= E^B\int_0^t\int_{\mathbb{R}^d} u_0(B^x_s)\exp(W(g^B_{s,x}))\varphi(x) \\ &\qquad \times \biggl[Q(x,x)Hs^{2H-1} \\ &\qquad + H(2H-1)\int_0^s (Q(B^x_{s-r},x) \\ &\qquad - Q(x,x))r^{2H-2}\,dr\biggr]\,ds\,dx. \end{aligned} \tag{5.12}$$

The convergence in L^2 to zero of V_ε^2 follows from (5.11) and (5.12). ☐

6. Skorohod type equation and Chaos expansion. In this section, we consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + u \diamond \dfrac{\partial W}{\partial t}(t,x), & t\ge 0, x\in\mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{6.1}$$

The difference between the above equation and equation (1.1) is that here we use the Wick product $\diamond$. This equation is studied in Hu and Nualart [3] for the case $H_1=\cdots=H_d=\frac{1}{2}$, and in [4] for the case $H_1,\dots,H_d\in(\frac{1}{2},1)$, $2H_0+H_1+\cdots+H_d>d+1$. As in that paper, we can define the following notion of solution.

DEFINITION 6.1. An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that $E(u^2(t,x)) < \infty$ for all (t,x) is a (mild) solution to equation (6.1) if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s),\ s \geq 0,\ y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds

$$u(t,x) = p_t f(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)\delta W_{s,y}, \tag{6.2}$$

where $p_t(x)$ denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)\,dy$.

From [3], we know that the solution to equation (6.1) exists with an explicit Wiener chaos expansion if and only if the Wiener chaos expansion converges. Note that $g_{t,x}^B(r,z) := \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,B_{t-r}^x]}(z) \in \mathcal{H}$. Formally, we can write $g_{t,x}^B(r,z) = \delta(B_{t-r}^x - z)$ and we have

$$\int_0^t W(dr, B_{s-r}^x) = W(g_{t,x}^B) = \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - z)W(dr,z)\,dz.$$

Then in the same way as in Section 8 in [4] we can check that $u(t,x)$ given by (6.3) below has the suitable Wiener chaos expansion, which has to be convergent because $u(t,x)$ is square integrable. We state it as the following theorem.

THEOREM 6.2. *Suppose $H > \frac{1}{2} - \frac{1}{4}\gamma$ and u_0 is a bounded measurable function. Then the unique (mild) solution to equation (6.1) is given by the process*

$$u(t,x) = E^B[u_0(B_t^x)\exp(W(g_{t,x}^B) - \tfrac{1}{2}\|g_{t,x}^B\|_{\mathcal{H}}^2)]. \tag{6.3}$$

REMARK 6.3. We can also obtain a Feynman–Kac formula for the coefficients of the chaos expansion of the solution to equation (1.1)

$$u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(t,x))$$

with

$$h_n(t,x) = E^B[u_0(B_t^x)g_{t,x}^B(r_1,z_1)\cdots g_{t,x}^B(r_n,z_n)\exp(\tfrac{1}{2}\|g_{t,x}^B\|_{\mathcal{H}}^2)].$$

APPENDIX

In this section, we denote by $B^H = \{B_t^H, t \in \mathbb{R}\}$ a mean zero Gaussian process with covariance $E(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. Denote by $\mathcal{E}$ the space of all step functions on $[-T,T]$. On $\mathcal{E}$, we introduce the following scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}_0} = R_H(t,s)$, where if $t < 0$ we assume that $\mathbf{1}_{[0,t]} = -\mathbf{1}_{[t,0]}$. Let $\mathcal{H}_0$ be the closure of $\mathcal{E}$ with respect to the above scalar product.

For $r>0$, $\varepsilon>0$ and $\beta>0$, let

$$f^{\varepsilon}(r):=\frac{1}{4\varepsilon^2}[2r^{\beta}-|r-2\varepsilon|^{\beta}-(r+2\varepsilon)^{\beta}].$$

It is easy to see that

$$\lim_{\varepsilon\downarrow 0} f^{\varepsilon}(r)=\beta(\beta-1)r^{\beta-2}. \tag{A.1}$$

Lemma A.1. *For any $r>0$, $\varepsilon>0$ and $0<\beta<2$,*

$$|f^{\varepsilon}(r)|\le 64r^{\beta-2}. \tag{A.2}$$

Proof. If $0<r<4\varepsilon$, then $|r-2\varepsilon|^{\beta}<(2\varepsilon)^{\beta}$, $(r+2\varepsilon)^{\beta}<(6\varepsilon)^{\beta}$, and hence (noting that $\beta<2$)

$$|f^{\varepsilon}(r)|\le 4^{\beta+1}\varepsilon^{\beta-2}\le 64r^{\beta-2}.$$

On the other hand, if $r\ge 4\varepsilon$, then

$$r^{\beta}-|r-2\varepsilon|^{\beta}=2\varepsilon\beta\int_0^1(r-2\lambda\varepsilon)^{\beta-1}\,d\lambda,$$

$$r^{\beta}-(r+2\varepsilon)^{\beta}=-2\varepsilon\beta\int_0^1(r+2\lambda\varepsilon)^{\beta-1}\,d\lambda,$$

and hence

$$\begin{aligned} f^{\varepsilon}(r)&=\frac{1}{2\varepsilon}\beta\int_0^1[(r-2\lambda\varepsilon)^{\beta-1}-(r+2\lambda\varepsilon)^{\beta-1}]\,d\lambda\\ &=2\beta(\beta-1)\int_0^1\int_0^1\lambda(r-2\lambda\varepsilon+4\mu\lambda\varepsilon)^{\beta-2}\,d\mu\,d\lambda. \end{aligned}$$

Therefore, using $\beta<2$ and $r\ge 4\varepsilon$ we obtain

$$|f^{\varepsilon}(r)|\le 2\beta(r-2\varepsilon)^{\beta-2}\le 4r^{\beta-2}\left(\frac{r-2\varepsilon}{r}\right)^{\beta-2}\le 16r^{\beta-2}.$$ □

Lemma A.2. *For any $s>0$, $0<\beta<1$ and $\phi\in C^{\alpha}([0,T])$ with $\alpha>1-\beta$,*

$$\lim_{\varepsilon\to 0}\int_0^s\phi(r)f^{\varepsilon}(r)\,dr=\phi(0)\beta s^{\beta-1}+\beta(\beta-1)\int_0^s(\phi(r)-\phi(0))r^{\beta-2}\,dr. \tag{A.3}$$

Moreover,

$$\left|\int_0^s\phi(r)f^{\varepsilon}(r)\,dr\right|\le C(\beta,\alpha)(\|\phi\|_{\infty}s^{\beta-1}+\|\phi\|_{\alpha}s^{\alpha+\beta-1}). \tag{A.4}$$

Proof. The lemma follows easily from (A.5) and (A.2) if we rewrite

$$\int_0^s\phi(r)f^{\varepsilon}(r)\,dr=\phi(0)\int_0^s f^{\varepsilon}(r)\,dr+\int_0^s[\phi(r)-\phi(0)]f^{\varepsilon}(r)\,dr.$$ □

LEMMA A.3. *For any bounded function $\phi \in \mathcal{H}_0$ and any $s,t \ge 0$, we have*

$$\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}_0} = H\int_0^s \phi(r)[r^{2H-1} + \operatorname{sign}(t-r)|t-r|^{2H-1}]\,dr. \tag{A.5}$$

If $u < s < t$, we have

$$\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[u,t]}\rangle_{\mathcal{H}_0} = H\int_0^s \phi(r)[(t-r)^{2H-1} - \operatorname{sign}(u-r)|u-r|^{2H-1}]\,dr. \tag{A.6}$$

PROOF. We only have to prove (A.5) since (A.6) follows easily. Without loss of generality, assume that $\phi = \sum_{i=1}^n a_i \mathbf{1}_{[t_{i-1},t_i]}$, where $0 = t_0 \le t_1 \le \cdots \le t_n = s$. (If $t < s$, we assume that $t = t_i$ for some $0 < i < n$.) Then

$$\begin{aligned}
\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}_0} &= E\sum_{i=1}^n a_i(B^H_{t_i} - B^H_{t_{i-1}})B^H_t \\
&= \sum_{i=1}^n a_i \frac{1}{2}(t_i^{2H} - t_{i-1}^{2H} + |t-t_{i-1}|^{2H} - |t-t_i|^{2H}) \\
&= H\int_0^s \phi(r)[r^{2H-1} + \operatorname{sign}(t-r)|t-r|^{2H-1}]\,dr. \qquad \square
\end{aligned}$$

Using Lemma A.3 and similar arguments to those in the proof of Lemma A.2, we can prove the following lemma.

LEMMA A.4. *For any $s > 0$, for any $\phi \in C^\alpha([0,T])$ with $\alpha > 1-2H$,*

$$\lim_{\varepsilon\to 0}\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0} = \phi(s)Hs^{2H-1} + c_0\int_0^s (\phi(s-r) - \phi(s))r^{2H-2}\,dr,$$

where $c_0 = H(2H-1)$. Moreover,

$$\left|\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0}\right| \le C(H,\alpha)(\|\phi\|_\infty s^{2H-1} + \|\phi\|_\alpha s^{\alpha+2H-1}). \tag{A.7}$$

PROOF. Applying Lemma A.3 and making a substitution, we get

$$\begin{aligned}
&\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0} \\
&\quad = \frac{H}{2\varepsilon}\int_0^s \phi(s-u)[(u+\varepsilon)^{2H-1} - \operatorname{sign}(u-\varepsilon)|u-\varepsilon|^{2H-1}]\,du \\
&\quad =: H\phi(s)\int_0^s g^\varepsilon(u)\,du + H\int_0^s [\phi(s-u) - \phi(s)]g^\varepsilon(u)\,du,
\end{aligned}$$

where we let

$$g^\varepsilon(u) = \frac{1}{2\varepsilon}[(u+\varepsilon)^{2H-1} - \operatorname{sign}(u-\varepsilon)|u-\varepsilon|^{2H-1}].$$

If $0<u<2\varepsilon$, we have $|g^{\varepsilon}(u)|\le 16r^{2H-2}$. On the other hand, if $u>2\varepsilon$,

$$|g^{\varepsilon}(u)| = \left|\frac{1}{2\varepsilon}[(u-\varepsilon)^{2H-1}-(u+\varepsilon)^{2H-1}]\right|$$
$$= \frac{1}{2}(1-2H)\int_{-1}^{1}(u-\lambda\varepsilon)^{2H-2}\,d\lambda \le (1-2H)u^{2H-2}.$$

Then the lemma follows by noticing that $\lim_{\varepsilon\to 0} g^{\varepsilon}(u) = (2H-1)u^{2H-2}$. □

LEMMA A.5. *For any $\phi\in C^{\alpha}([0,T])$ with $\alpha>1-2H$, for any $s>0$,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left\langle \frac{1}{2\varepsilon}\int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}\phi(\theta)\,d\theta, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0}\\ &\qquad = \phi(s)Hs^{2H-1} + H(2H-1)\int_0^s(\phi(s-r)-\phi(s))r^{2H-2}\,dr.\end{aligned}\tag{A.8}$$

Moreover,

$$\begin{aligned}&\left|\left\langle \frac{1}{2\varepsilon}\int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}\phi(\theta)\,d\theta, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0}\right|\\ &\qquad \le C(H,\alpha)(\|\phi\|_{\infty}Hs^{2H-1} + \|\phi\|_{\alpha}s^{\alpha+2H-1}).\end{aligned}\tag{A.9}$$

PROOF. By Fubini's theorem and making a substitution, we have

$$\begin{aligned}&\left\langle \frac{1}{2\varepsilon}\int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}\phi(\theta)\,d\theta, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}_0}\\ &\qquad = \frac{1}{4\varepsilon^2}E\left[\int_0^s \phi(\theta)(B^H_{\theta+\varepsilon}-B^H_{\theta-\varepsilon})(B^H_{s+\varepsilon}-B^H_{s-\varepsilon})\,d\theta\right]\\ &\qquad = \frac{1}{8\varepsilon^2}\int_0^s \phi(s-\theta)[2r^{2H}-|r-2\varepsilon|^{2H}-(r+2\varepsilon)^{2H}]\,dr.\end{aligned}$$

Then (A.8) and (A.9) follow from Lemma A.2. □

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